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Dynamic conceptualisation of density of $\mathbb{Q}$ in $\mathbb{R}$: Upper-secondary school students' spontaneous conceptions and arguments

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While research in Mathematics Education has mainly focused on density of $\mathbb{Q}$, little is known about density of $\mathbb{Q}$ in $\mathbb{R}$, despite its importance in grasping fundamental Calculus concepts such as continuity. This study investigates upper-secondary school students' spontaneous conceptions and arguments regarding the density of $\mathbb{Q}$ in $\mathbb{R}$. A classroom activity and group interviews focused on continued fractions, which provide a unified representation of rational and irrational numbers, were carried out. The data consist of the students' written and spoken statements, and the data analysis strategy follows a thematic analysis approach with some specific modifications. The research findings indicate that students' spontaneous conceptions are dynamic processes that range over different categories supported by different arguments and warrants.

Keywords: Density of $\mathbb{Q}$ in $\mathbb{R}$, spontaneous conceptions, arguments, continued fractions, warrant.

Rationale

The concept of density of $\mathbb{Q}$ in $\mathbb{R}$ is crucial because it ensures that irrational numbers can be conceived of as limits of sequences of rational numbers, guaranteeing that each point of the real line corresponds to a number. Students encounter the concept of density already when dealing with rational numbers, but while the set of rational numbers is dense, the set of real numbers is both dense and complete. If the density of $\mathbb{Q}$ becomes an intuitive model for the density of $\mathbb{Q}$ in $\mathbb{R}$, students may struggle to grasp the difference between $\mathbb{Q}$ and $\mathbb{R}$ and the role of completeness for $\mathbb{R}$ as the ideal domain for central Calculus phenomena such as continuity and completeness.

As research shows (Branchetti & Durand-Guerrier, 2018; McMullen & Van Hoof, 2020; Putra, 2019), exploring the transitions between numerical sets is crucial to avoid obstacles in future mathematical learning and the density-concept is crucial in this sense. Nevertheless, according to the literature, while the concept of density of $\mathbb{Q}$ is widely investigated (e.g., Hannula et al., 2012; Kim & Kwon, 2019), the density in the transition from $\mathbb{Q}$ to $\mathbb{R}$ – the one of density of $\mathbb{Q}$ in $\mathbb{R}$ – has received little attention from research in Mathematics Education (Marmur et al., 2020). Marmur et al. (2020) investigate how first-year mathematics undergraduates grasp the concept of density of $\mathbb{Q}$ in $\mathbb{R}$. However, students face this concept, at least intuitively, already in upper-secondary school. In this context, spontaneous conceptions, intended as intuitive beliefs formed from experiences in various contexts, including everyday life (Fujii, 2014), serve as a foundation for future institutionalised concepts. This paper, based on the first author's master's thesis (Grisendi, 2024), examines the spontaneous conceptions of the density of $\mathbb{Q}$ in $\mathbb{R}$ of two classes of Italian upper-secondary school students (aged 17/18). The students' spontaneous conceptions, the kind of arguments and justifications they use to support their reasoning in working on this concept, are categorised and integrated, focusing on the following research questions: (1) What are students' spontaneous conceptions regarding the concept of density of $\mathbb{Q}$ in $\mathbb{R}$, and what arguments do they use to support them? (2) How do these arguments and spontaneous conceptions support students' reasoning processes as they evolve during a classroom activity focused on the concept of density of $\mathbb{Q}$ in $\mathbb{R}$?

Theoretical framework

The theoretical framework is composed of two elements: Balacheff's (1988) classification of different approaches to proof and Toulmin's (1958) concept of warrant. These two components are first introduced, and then their interrelations in the present investigation are discussed.

Balacheff (1988) distinguishes between *pragmatic proof* (based on actions) and *intellectual proof* (which employs verbalisations of the properties of objects and their relationships). Additionally, he identifies a taxonomy for the types of proof conceived as steps between *pragmatic proof* and *intellectual proof*, characterised by increasing sophistication: *naïve empiricism* (examples proposed are considered self-evident); *crucial experiment* (a paradigmatic example is used, suggesting that if it holds in one case, it holds in all cases); *generic example* and *thought experiment* (an example is used as a "support," but the reasoning is general).

According to Toulmin (1958), an argument has three constituent parts: claim, datum, and warrant. The claim refers to the main assertion one is attempting to prove; the datum includes the reasons, evidence, facts, and/or information that support the claim; the warrant represents the logical connection linking the datum to the claim supporting the provided argument. Our analysis focuses specifically on the warrant; instead, the claim and the datum are provided in the task and do not vary.

Balacheff's (1988) classification of different approaches to proof accounts for what students consider a reliable validation and the soundness of their inference and Toulmin's (1958) idea of warrant provides critical insights into the type of justification used to support the validation. In this sense, the two components mentioned above collectively support the analysis of student conceptions and arguments.

Research method

Our research is guided by an *ad hoc* constructed strategy based on different investigation themes. These themes include students' general arguments, with a specific focus on students' mathematical arguments, and finally, students' conceptions. To stimulate the emergence of spontaneous conceptions, an activity focused on continued fractions, which facilitates the transition from the countable ($\mathbb{Q}$) to the continuous ($\mathbb{R}$), was designed. Two classes for a total of 42 upper-secondary school students were involved. The activity, divided into two parts, was conducted under the lead of the first author in the presence of the classroom teachers but without their intervention. In the first part (two hours), students respond anonymously, working in small groups. This part involves a worksheet, which begins with a brief contextualised introduction to continued fractions, where a hypothetical student proposes a continued fraction as an approximation of an irrational solution of a second-grade equation; then, students are asked to argue for or against the statement: "[...] given any real number, can I always find a rational number that is as close to it as I want?" After reading the initial part of the worksheet, students complete two parts, called Assignment 1 and Assignment 2. In Assignment 1, students write down a points list of arguments for or against the statement read before, while in Assignment 2, starting from the generated list, students are asked to produce a discursive response by writing an argumentative text. These tasks were chosen to track students' reasoning processes and identify the kinds of arguments they use to support their conceptions. Finally, students are interviewed by the researcher a few days later via semistructured group interviews, inviting them to deepen the arguments presented in their group. In this sense, the collected data were written or spoken sentences produced by the students while trying to argue their position concerning the task-claim.

The data analysis strategy used in this research partially overlaps with the thematic analysis described in Xu and Zammit (2020). However, it differs from the latter in several aspects, as it was constructed ad hoc for the data decoding process. Below, we explain our approach, showing analogies and differences to the thematic analysis approach. As revealed before, an *ad hoc* strategy was constructed because of the data set's complexity. We designed a structured data analysis process, i.e., a flowchart, that guides the coding of the data through structured questions. Thanks to this tool, some of the thematic analysis steps are unnecessary for our analysis.

The first step of our data analysis corresponds with the first step of the thematic analysis. We reviewed the students' sheets and prepared targeted questions for each group to explore their reasoning further. These questions are used to guide the initial part of the semistructured interviews. Indeed, only one question was repeated identically for every group: "How can you define, in your own words, a dense set?"

Also, our second step is analogous to the second step of thematic analysis (Xu and Zammit, 2020). We constructed a category structure, the theory-driven code, developed through both deductive (drawing from the relevant literature) and inductive (drawing from the data) approaches (Mayring, 2015), and we elaborated the above-mentioned flowchart as the data-driven code to simplify the process. The flowchart guided the classification process of the statements through structured questions. Following the flowchart's questions multiple times with the same students' statement, we explored its possibly different interpretations. This flexibility enabled us to observe a single statement considering different aspects of students' conceptions and arguments.

Below, we describe the category structure. Initially, we focused on classifying the types of arguments students proposed. Deductively, we referred to categories identified in the section Theoretical framework; inductively, we identified the remaining warrant-categories, supposing that each has a dual category. In this sense, a distinction was made between *pragmatic warrant* (related to utility and efficiency) and *theoretical warrant* (related to properties and definitions). Subsequently, for each of Inglis' (2007) warrant-categories – *inductive warrant* (the use of a certain number of examples), *intuitive warrant* (the use of observations or examples that reflect some underlying cognitive structure), and *deductive warrant* (the use of deduction from axioms, algebraic manipulations, and counterexamples), a distinction in *operational warrant* (related to a procedure) and *structural warrant* (related to properties and relationships between objects) was made. Finally, we categorised the spontaneous conceptions of density of $\mathbb{Q}$ in $\mathbb{R}$ that emerged during the activity. These conceptions were categorised inductively into two types: *analytic conception* (based on a fundamental idea of sequences, intuitively linked to the construction of the set $\mathbb{R}$ through Cauchy sequences), *synthetic* *conception* (based on geometric-spatial concepts and related to the number line, intuitively linked to the construction of the set ℝ through Dedekind cuts).

The following three steps of thematic analysis refer to using a thematic map developed in the third step, revised and refined in the fourth step, and organised "into a coherent and consistent account" (Xu & Zammit, 2020, p. 7) in the fifth step. Because our analysis develops by referring to the flowchart, the process is structured and makes the analysis shorter. Indeed, thanks to the data analysis thematic code, immediately after classifying a statement in the category structure and tracking how students move towards a new statement, we can reconstruct and represent the group's reasoning through specific kinds of thematic maps, called schema and graphs. In the following section, we show two different kinds of maps used to represent the groups' reasoning.

The last step of thematic analysis coincides with this paper's Discussion and conclusion section.

Results

Below, we provide a summary analysis of just one of the twelve groups examined in the thesis, suitable to exemplify the methodology of data analysis developed and used to this end. We focus on Group 10, whose responses are particularly interesting regarding the reasoning process the students proposed. Before Group 10's detailed analysis, we present two examples of the thematic maps used in our research. These maps are designed to represent "the meanings in the data set as a whole" (Xu & Zammit, 2020, p. 6) and "to fit each theme into a broader overall story about the data" (Xu & Zammit, 2020, p. 7), effectively facilitating the visualisation of data analysis for each group.

Figure 1 shows the schema that presents the analysed categories for Group 10's spontaneous conceptions and arguments. Each category is represented by a box with a distinct shape and colour. The coloured rectangles provide information on the context of data collection (group number, Assignment 1, Assignment 2, Interview).

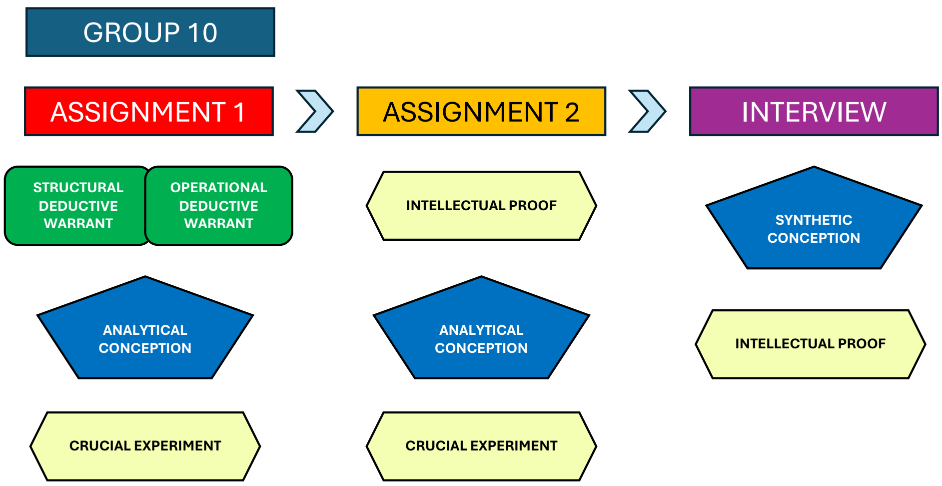

Figure 1: Schema of Group 10 students' arguments and spontaneous conceptions

Moreover, Group 10 students' reasoning can be effectively represented by a graph (Figure 2).

In the graph in Figure 2, coloured circles indicate the vertices, while dark blue arrows represent the edges. The logic behind the connections between the two statements is as follows: (1) If the edge between two statements, *i* and *i*+1, is a head-tail arrow, it indicates a temporal sequence (i.e., the arrow's tail represents the statement at time *i*, while the head represents the statement at time *i*+1); (2) If the edge is a head-head arrow, the two statements are semantically similar, as they reflect analogous reasoning expressed at different moments, possibly with variations in wording. The students' statements are displayed in light blue rectangles; the coloured circles indicate when the statement was made during the activity, and the coloured triangles indicate the category in which the statement was classified.

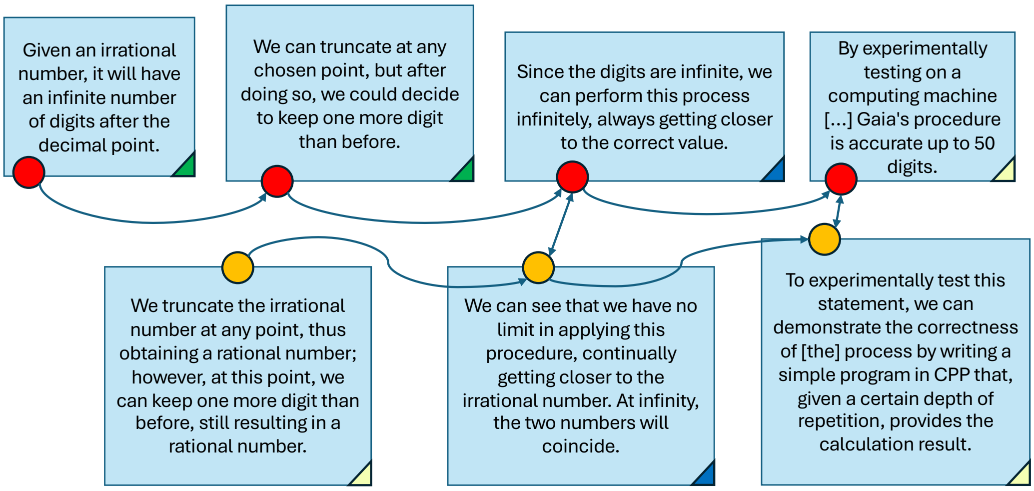

Figure 2: Graph of Group 10 students' arguments and spontaneous conceptions

With the support of the two previously presented schemas, we analyse Group 10's data collected. The students of this group initially employed a *deductive warrant* (Assignment 1). It first provided them with a theoretical reference (*structural deductive warrant*). Indeed, the statement, *Given an irrational number, it will have an infinite number of digits after the decimal point* (Figure 2), represents a fundamental property of irrational numbers. Additionally, an operational process (*operational deductive warrant*) was the foundation of their reasoning. Indeed, this is evident in the statement: *We can truncate* [the irrational number mentioned before] *at any chosen point, but after doing so, we could decide to keep one more digit than before* (Figure 2), which outlines a specific procedure.

Subsequently, in Assignment 2, Group 10 demonstrated a natural progression from a *theoretical warrant* (of which the *deductive warrant* is a specific case) to an instrument for justifying an *intellectual proof*. The statement classified as an *intellectual proof* is the following: *We truncate the irrational number at any point, thus obtaining a rational number; however, at this point, we can keep one more digit than before, still resulting in a rational number* (Figure 2). We can observe that, while in the *theoretical warrant* it is possible to distinguish between the reference to the theoretical concept and the proposal of an operational procedure, in the *intellectual proof*, the two components are no longer distinctly separable, producing a unified and coherent reasoning process.

It is evident that Group 10 is aware of the problem of generalisation. To validate their reasoning, the students conclude their analysis by providing a paradigmatic example, specifically a CPP program. Since the program is designed to generalise the process, Group 10's statements regarding this program are considered in the *crucial experiment* category.

The procedure proposed by Group 10 is further supported by the *analytical conception* that the students have regarding the concept of the density of ℚ in ℝ. In Assignment 1, the statement – *Since the digits are infinite, we can perform this process infinitely, always getting closer to the correct value* (Figure 2) – suggests a conception based on the image of a sequence of rational numbers obtained by the "truncation" process, converging towards the "*correct value*". In Assignment 2, the statement – *We can see that we have no limit in applying this procedure, continually getting closer to the irrational number. At infinity, the two numbers will coincide* (Figure 2) – intuitively expresses the idea of a sequence of rational numbers as a single number that "transforms" over time.

During the Interview, the *analytical conception* evolves into a *synthetic* one. Student A's statement – *There is a value at every point.* […] *You have a value for every point; you can always divide a point in half* (Figure 2) – explicitly refers to spatial and geometric concepts, as indicated by the terms "value *at every point,*" "value *for every point*," and "*divide a point* in half," which imply a correspondence between ℝ and the number line. Furthermore, student B's statement – *I know that the cardinality of natural numbers and rational numbers is the same; whereas, for real numbers, it is different.* […] *It is larger. Between natural and rational numbers, we can find a "correlation," so we can say they have the same cardinality. However, real numbers are dense, meaning there is no space between one number and another* (Figure 2) – explicitly refers to a property of ℝ set, namely the cardinality of the continuum. This becomes clear when student B states that the cardinality of ℝ set is "*different*" from that of ℕ and ℚ because it is "*larger*." This statement is classified as *intellectual proof* since the property of ℚ is used to argue that ℚ is a dense set. Moreover, it supports the emergence of the *synthetic conception*, as indicated by the terms "there is no *space* between one number and another".

Discussion and conclusion

In the Rationale section, we pointed out that only one of the referenced studies specifically addresses the concept of the density of ℚ in ℝ: the research conducted by Marmur et al. (2020). Marmur et al.'s (2020) findings are twofold. Firstly, the findings demonstrate "unconventional student understandings" (Marmur et al., 2020, p. 7) of the concept of density of ℚ in ℝ, e.g., some students use the concept of infinity only as a process, as authors write: "attend to the process of creating infinitely many rational numbers [...], rather than to the existence of these numbers" (authors' italics) (Marmur et al., 2020, p. 5). Secondly, rather than attending to general reasoning, students concentrate "on specific examples, contexts, processes, and personal meanings, consequently reducing the intended abstraction level of the task" (Marmur et al., 2020, p. 7). These authors' study focuses on university students, while our research participants are upper-secondary students. As the participants' linguistic and cognitive tools in the two investigations differ, the research findings differ, as detailed below.

On the one hand, according to our analysis, the students' understanding cannot be classified as "unconventional," as highlighted by Marmur et al. (2020). In our research, each group presents a different pathway and the term "unconventional" loses its sense. In particular, the groups that produced a less comprehensive conceptualisation tended to rely on categories of argument and proof that did not require intricate sentence structures or advanced reasoning (such as *crucial experiment*). Moreover, their spontaneous conceptions reflected only the *analytical conception*, likely because it aligns more closely with the thesis presented in the worksheet, guiding their reasoning. Addressing the first research question, it becomes evident that the *analytical conception* is perceived as more intuitive and, therefore, primary compared to the *synthetic conception*. However, the coexistence of both conceptions ultimately facilitates significant advancements in the conceptual understanding of density of $\mathbb{Q}$ in $\mathbb{R}$.

On the other hand, our results shed light on when and why students need to refer to different kinds of warrants, e.g., a specific practical example or, instead, some intuitive properties of a mathematical object. So, we recognised that using examples is only sometimes about reducing abstraction, as suggested by Hazzan (1999); but, in some cases, students feel obligated to present an example, even if their reasoning is general because they do not have suitable linguistic tools to express their ideas in another way. This leads us to the second research question. We can state that the students' reasoning varies across different groups, each beginning from distinct starting points.

Moreover, groups capable of navigating the various typologies of arguments and proofs demonstrate a more in-depth reasoning process. It is important to note that some elements of the dual categories of warrants (e.g., the *intuitive theoretical warrant*) never appeared in the data. This is likely because these categories require cognitive and linguistic tools that are not yet fully accessible to upper-secondary students. During the written phases of the activity (Assignment 1 and Assignment 2), almost all categories of arguments were explored at least once in the students' responses. However, during the Interview, no categories related to warrant typologies emerged. This discrepancy may be attributed to the fact that the cognitive and linguistic tools available to students are less developed in oral communication than in written tasks. Additionally, during the written phases, students had the opportunity to collaborate in groups, allowing for the emergence of more complex and well-articulated arguments.

Ultimately, the data analysis reveals that students' spontaneous conceptions are dynamic and evolve during the activity. The categories employed in this analysis help identify connections between different conceptions and the arguments that connect them. A potential future research development is extending the investigation, using the developed methodology, to concepts contiguous to the one being examined, such as continuity.

References

Balacheff, N. (1988). A study of students' proving processes at the junior high school level. In I. Wirszup & R. Streit (Eds.), *Proceedings of the Second UCSMP International Conference on Mathematics Education* (Vol. 2, pp. 284–298). NCTM.

Branchetti, L., & Durand-Guerrier, V. (2018). Are all denumerable sets of numbers order-isomorphic? In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N.M. Hogstad (Eds.), *Proceedings of INDRUM 2018* (pp. 14–23). University of Agder and INDRUM.

Fujii, T. (2014). Misconceptions and Alternative Conceptions in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 453–455). Springer. https://doi.org/10.1007/978-94-007-4978-8 114

Grisendi, C. (2024). *Un'indagine sulla concettualizzazione dinamica della densità di Q in R. Analisi delle concezioni spontanee e delle argomentazioni degli studenti di Scuola Secondaria di Secondo Grado* [Master's thesis, the University of Modena and Reggio Emilia]. MoReThesis - Unimore. https://morethesis.unimore.it/theses/available/etd-06272024-191214/

Hannula, M. S., Laine, A., Penhkonen, E., & Kaasila, R. (2012). Learning density of numbers in elementary teacher education. In G.H. Gunnarsdóttir, F. Hreinsdóttir, G. Pálsdóttir, M.S. Hannula, M. Hannula-Sormunen, & E. Jablonka (Eds.), *Proceedings of NORMA 11 Nordic Conference on Mathematics Education, May 11-14, 2011, Reykjavìk, Iceland* (pp. 1–10). University of Iceland Press.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics, 40*(1), 71–90. https://doi.org/10.1023/A:1003780613628

Inglis, M., Mejia-Ramos, J.P., & Simpson, A. (2007). Modelling mathematical argumentation: the importance of qualification. *Educational Studies in Mathematics, 66*(1), 3–21. https://doi.org/10.1007/s10649-006-9059-8

McMullen, J., & Van Hoof, J. (2020). The role of rational number density knowledge in mathematical development. *Learning and Instruction, 65*, Article 101228. https://doi.org/10.1016/j.learninstruc.2019.101228

Marmur, O., Moutinho, I., & Zazkis, R. (2020). On the density of Q in R: Imaginary dialogues scripted by undergraduate students. *International Journal of Mathematical Education in Science and Technology, 53*(6), 1297–1325. https://doi.org/10.1080/0020739X.2020.1815880

Mayring, P. (2015*). Qualitative Content Analysis: Theoretical Background and Procedures.* In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to Qualitative Research in Mathematics Education - Examples of Methodology and Methods* (pp. 365–380). Springer. https://doi.org/10.1007/978-94-017-9181-6_13

Kim, D., & Kwon, O. N. (2019). Moving fingers and the density of rational numbers: An inclusive materialist approach to digital technology in the classroom. In P. Drijvers, K. M. Clark, & S. Rezat (Eds.), *Proceedings of CERME11* (pp. 2868–2875). Utrecht University and ERME. https://hal.science/hal-02428258/

Putra, Z. H. (2019). Praxeological Change and the Density of Rational Numbers: The Case of Pre-service Teachers in Denmark and Indonesia. *EURASIA Journal of Mathematics, Science and Technology Education, 15*(5), Article em1711. https://doi.org/10.29333/ejmste/105867

Toulmin, S. E. (1958). *The uses of argument*. Cambridge University Press.

Xu, W., & Zammit, K. (2020). Applying Thematic Analysis to Education: A Hybrid Approach to Interpreting Data in Practitioner Research. *International Journal of Qualitative Methods, 19*, 1–9. https://doi.org/10.1177/1609406920918810